Int. J. Heat Mass Transfer. Vol. 19, pp. 1121–1125. Pergamon Press 1976. Printed in Great Britain

LAMINAR FLOW NATURAL CONVECTION FROM THE OPEN VERTICAL CYLINDER WITH UNIFORM HEAT FLUX AT THE WALL

H. J. HETHERINGTON* and T. D. PATTEN†

(*Received* 31 *March* 1976)

Abstract—A solution is obtained for fully developed flow of a constant-property fluid, with the lowest temperature equal to the fluid bulk temperature. This is satisfactory only for low values of Rayleigh number, up to about 20, that is, for relatively long cylinders.

A second solution has in addition a thermal entry region, for application to shorter cylinders, to the point where the entry region fills the cylinder. This solution has not yet been made mathematically exact. Nusselt numbers up to the terminal point, at Rayleigh number of about 16 000, are identical with those for the finite difference result of Dyer [6].

NOMENCLATURE

a, cylinder radius;
c, specific heat;
d, diameter;
e, axial temperature gradient;
g, gravitational acceleration;
h, overall heat-transfer coefficient $= \dfrac{q}{\theta}$;
k, thermal conductivity;
l, cylinder height;
p, pressure;
q, heat flux at wall;
r, radial co-ordinate;
t, temperature, measured from fluid bulk temperature;
u, velocity in x-direction;
x, axial co-ordinate;
D, derivative;
V, constant in equation (14);
Q_W, total heat flow across walls;
Q_E, total heat supply at the entry plane.

Greek symbols

α, thermal diffusivity, $= \dfrac{k}{\rho c}$;
β, volume thermal expansivity;
θ, mean temperature difference between cylinder and fluid bulk;
μ, dynamic viscosity;
ν, kinematic viscosity, $= \dfrac{\mu}{\rho}$;
ρ, mass density;
ϕ, radial temperature component;
κ, constant in equation (15).

Dimensionless quantities

R, $= \dfrac{r}{a}$, radial co-ordinate;
Λ, $= \left(\dfrac{g\beta e a^4}{\nu\alpha}\right)$, form parameter;
U, $= \dfrac{u}{u_{\max}}$, axial velocity;
Φ, $= \dfrac{\phi}{\phi_{\max}}$, radial temperature;
Θ, $= \dfrac{\theta}{2\kappa}$, temperature difference;
E, $= \dfrac{Q_E}{Q_W}$, heat supply in entry region;
X, $= \dfrac{x}{El}$, axial co-ordinate;
T, $= \dfrac{t}{\phi_{\max}}$, conventional temperature;
Gr, $= \dfrac{g\beta q d^5}{k\nu^2 l}$, Grashof number;
Nu, $= \dfrac{qd}{\theta k}$, Nusselt number;
Re, $= \dfrac{\bar{u}d^2}{\nu l}$, Reynolds number;
Pr, $= \dfrac{\nu}{\alpha}$, Prandtl number;
Ra, $= GrPr$, Rayleigh number;
Pe, $= RePr$, Péclet number.

*Lecturer in Mechanical Engineering, The University, Dundee, DD1 4HN, Scotland.

†Professor of Mechanical Engineering, Heriot-Watt University, Riccarton, Currie, Edinburgh, EH14 4AS, Scotland.

INTRODUCTION

THE SYSTEM treated is a vertical cylinder open at both ends to a large free volume of essentially isothermal fluid, with a heat flux at the cylinder walls.

In a very narrow cylinder, with parabolic velocity profile, fluid temperature differing very little from the wall temperature, and wall temperature increasing linearly with height, it is quite easily shown that

$$\bar{u} = \tfrac{1}{32}\frac{g\beta d^2}{\nu}\theta \tag{1}$$

$$q = \tfrac{1}{64}\frac{g\beta k d^3}{\nu\alpha}\theta^2. \tag{2}$$

These equations may also be written

$$Pe = \tfrac{1}{4}Ra^{\frac{1}{2}} \tag{3}$$

$$Nu = \tfrac{1}{8}Ra^{\frac{1}{2}}. \tag{4}$$

The parameters have the particular forms appropriate for the present system. These conditions are applicable only for small values of Ra, up to about 5.

A complete solution for natural convection in cylinders will cover a range of Rayleigh numbers from the long narrow cylinder, at low values, through shorter or wider cylinders, to the vertical plane wall, at high values of Ra. Some relations of this kind were presented by Elenbaas [1], including one for the isothermal cylinder. But due to analytical difficulties these were on a very empirical basis. Information is given for overall heat transfer, but very little on fluid velocities or temperatures.

Sparrow and Gregg [2] have given a similarity solution for the vertical wall with uniform heat flux. For the local convection coefficient h_x, at distance x from the foot of the wall, when $Pr = 0.7$,

$$\frac{h_x x}{k} = 0.48\left(\frac{g\beta q x^4}{k\nu^2}\right)^{\frac{1}{5}}. \tag{5}$$

Integration over the whole height l gives for the mean convection coefficient h

$$\frac{hl}{k} = 0.576\left(\frac{g\beta q l^4}{k\nu^2}\right)^{\frac{1}{5}}. \tag{6}$$

Transformation into Nu and Gr for the cylinder does not alter the constant 0.576, but it has to be divided by $Pr^{\frac{1}{5}} = 0.931$ to give

$$Nu = 0.619Ra^{\frac{1}{5}}. \tag{7}$$

Solutions for steady natural convection in a closed region with fully-developed flow and uniform heat generation were given by Woodrow [3], firstly between wide vertical plane walls, and secondly inside a vertical cylinder. The rectangular co-ordinate part of this work was adapted to the open-ended system without heat generation by Wordsworth [4]. In the present paper a similar adaptation is made of Woodrow's cylindrical co-ordinate solution.

THEORY FOR FULLY DEVELOPED FLOW

The idealisations are made that fluid density varies only in forming a buoyancy term, other properties do not change with temperature or pressure, viscous dissipation is negligible, velocity is zero except in the x-direction and pressure varies only with x, and the pressure difference between upper and lower cylinder ends equals the external hydrostatic pressure difference.

$$p_0 - p_l = g\rho l. \tag{8}$$

It is a property of the system that axial temperature gradient is constant, and the radial component ϕ is independent of x. Hence, if when

$$x = r = 0, \quad t = 0 \tag{9}$$

$$t = ex + \phi. \tag{10}$$

The equations of motion in the x-direction, conservation of energy, state of the fluid, and continuity are:

$$\frac{dp}{dx} + g\rho = \frac{\mu}{r}\left(\frac{du}{dr} + r\frac{d^2u}{dr^2}\right) \tag{11}$$

$$\rho c u\frac{\partial t}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial t}{\partial r}\right) + \frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right) \tag{12}$$

$$\rho = \rho_0(1-\beta t) \tag{13}$$

$$\int_0^a ru\,dr = a^2V \tag{14}$$

where V is a constant. Using some of the auxiliary relations, equation (11) is solved by separating the variables x and r. The constant κ and the dimensionless form parameter Λ are introduced, where

$$\kappa = \tfrac{1}{2}el \tag{15}$$

and

$$\Lambda^4 = \frac{g\beta e a^4}{\nu\alpha}. \tag{16}$$

Writing D for d/dr, the differential equation in u is obtained

$$\left[\left(D^2 + \frac{1}{R}D\right)^2 + \Lambda^4\right]u = 0. \tag{17}$$

Conditions for u are that u is an even function of r, and that $u = 0$ when $r = \pm a$.

The solutions for u, ϕ and Ra are in terms of two of the Kelvin functions namely ber and bei, and their derivatives, ber′ and bei′:

$$u = V\Lambda\frac{\operatorname{bei}\Lambda\operatorname{ber}\Lambda R - \operatorname{ber}\Lambda\operatorname{bei}\Lambda R}{\operatorname{ber}\Lambda\operatorname{ber}'\Lambda + \operatorname{bei}\Lambda\operatorname{bei}'\Lambda} \tag{18}$$

$$\phi = \frac{1}{4}\frac{\nu\alpha l}{g\beta a^4}\Lambda^4\frac{\operatorname{ber}\Lambda(\operatorname{ber}\Lambda R - 1) + \operatorname{bei}\Lambda\operatorname{bei}\Lambda R}{\operatorname{ber}\Lambda} \tag{19}$$

$$Ra = \frac{g\beta q d^5}{\nu\alpha k l} = 16\Lambda^5\frac{\operatorname{ber}\Lambda\operatorname{ber}'\Lambda + \operatorname{bei}\Lambda\operatorname{bei}'\Lambda}{\operatorname{ber}\Lambda}. \tag{20}$$

Using Bessel functions tables, the right side of equation (20) is plotted against Λ in Fig. 1. Only the

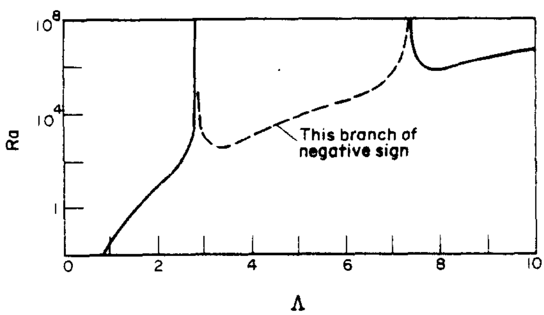

FIG. 1. Cylinder with fully developed temperatures. Rayleigh number against form parameter Λ.

first branch of the curve is used to cover all values of Rayleigh number, which is the only independent variable. Other dimensionless quantities are obtained as functions of ΛR and Λ, or Λ only:

$$U = \frac{\mathrm{bei}\,\Lambda\,\mathrm{ber}\,\Lambda R - \mathrm{ber}\,\Lambda\,\mathrm{bei}\,\Lambda R}{\mathrm{bei}\,\Lambda} \quad (21)$$

$$\Phi = \frac{\mathrm{ber}\,\Lambda\,\mathrm{ber}\,\Lambda R + \mathrm{bei}\,\Lambda\,\mathrm{bei}\,\Lambda R - \mathrm{ber}\,\Lambda}{\mathrm{ber}^2\,\Lambda + \mathrm{bei}^2\,\Lambda - \mathrm{ber}\,\Lambda} \quad (22)$$

$$\bar{U} = 2\,\frac{\mathrm{ber}\,\Lambda\,\mathrm{ber}'\,\Lambda + \mathrm{bei}\,\Lambda\,\mathrm{bei}'\,\Lambda}{\Lambda\,\mathrm{bei}\,\Lambda} \quad (23)$$

$$Nu = 2\Lambda\,\frac{\mathrm{ber}\,\Lambda\,\mathrm{ber}'\,\Lambda + \mathrm{bei}\,\Lambda\,\mathrm{bei}'\,\Lambda}{\mathrm{ber}^2\,\Lambda + \mathrm{bei}^2\,\Lambda} \quad (24)$$

$$Pe = 4\Lambda\,\frac{\mathrm{ber}\,\Lambda\,\mathrm{ber}'\,\Lambda + \mathrm{bei}\,\Lambda\,\mathrm{bei}'\,\Lambda}{\mathrm{ber}\,\Lambda}. \quad (25)$$

but as Ra increases, instead of approaching the vertical wall solution, it tends towards a maximum value of Nu of approximately 2.982.

It is conspicuous that the heat supply required at the entry plane of the cylinder, to establish the radial temperature profile, is not accounted for. This is termed Q_E, the heat flow rate across the cylinder walls Q_W, and the ratio of these two heat rates E. The dimensionless mean temperature difference Θ is also required.

$$Q_E = \int_0^a \rho c \phi u 2\pi r\,\mathrm{d}r \quad (26)$$

$$Q_W = \pi d l q \quad (27)$$

$$E = \frac{Q_E}{Q_W} = \frac{\Lambda\,\mathrm{bei}\,\Lambda(\mathrm{ber}^2\,\Lambda + \mathrm{bei}^2\,\Lambda - \mathrm{ber}\,\Lambda)}{4\,\mathrm{ber}\,\Lambda(\mathrm{ber}\,\Lambda\,\mathrm{ber}'\,\Lambda + \mathrm{bei}\,\Lambda\,\mathrm{bei}'\,\Lambda)} I_\Lambda \quad (28)$$

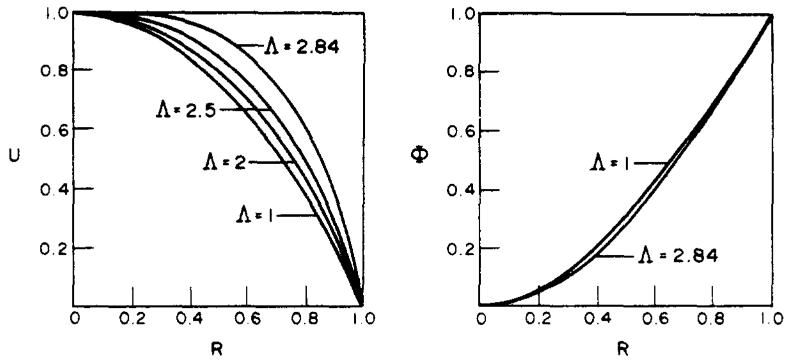

FIG. 2. Cylinder with fully developed temperatures. Upward velocity U, and radial temperature Φ, both against radius R, for some values of Λ.

Figure 2 shows there is not much variation in velocity U, and even less in radial temperature Φ, for the range of Λ used. Figure 3 shows that the present solution merges satisfactorily with that for the narrow cylinder,

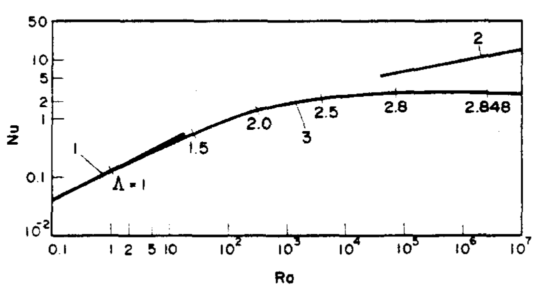

FIG. 3. Nusselt number against Rayleigh number: (1) Narrow cylinder: $Nu = 0.125Ra^{0.5}$; (2) Vertical wall: $Nu = 0.619Ra^{0.2}$; (3) Cylinder with fully developed temperatures, with some values of Λ.

where

$$I_\Lambda = \int_0^1 U\Phi R\,\mathrm{d}R \quad (29)$$

$$\Theta = \frac{\theta}{2\kappa} = \frac{\mathrm{ber}^2\,\Lambda + \mathrm{bei}^2\,\Lambda + \mathrm{ber}\,\Lambda}{4\,\mathrm{ber}\,\Lambda}. \quad (30)$$

It is seen from Table 1 that, with increasing Λ, E increases rapidly. As the entry plane heat supply Q_E increases in importance, the solution departs progressively from a proper representation of a relatively short cylinder.

CYLINDER WITH A THERMAL ENTRY REGION INCLUDED

The cylinder is now extended downward to allow the additional rate of heat supply Q_E across the walls. Velocities are unchanged, and, within the heated fluid, the radial temperature profile. In the thermal entry region, vertical temperature gradient is however not

Table 1. Results for overall heat transfer from vertical cylinders, for representative values of the form parameter Λ

Λ	Ra	Nu	Θ	I_Λ	E	Ra_1	Nu_1
1.0	1.020	1.218×10^{-1}	5.120×10^{-1}	9.80×10^{-2}	4.67×10^{-3}	1.015	1.221×10^{-1}
1.5	2.820×10^{1}	5.532×10^{-1}	5.650×10^{-1}	1.01×10^{-1}	2.57×10^{-2}	2.749×10^{1}	5.615×10^{-1}
2.0	3.549×10^{2}	1.380	7.520×10^{-1}	1.06×10^{-1}	1.00×10^{-1}	3.220×10^{2}	1.460
2.5	4.208×10^{3}	2.359	1.678	1.19×10^{-1}	4.74×10^{-1}	2.803×10^{3}	2.760
2.8	6.752×10^{4}	2.904	1.205×10^{1}	1.35×10^{-1}	4.79	1.166×10^{4}	3.900

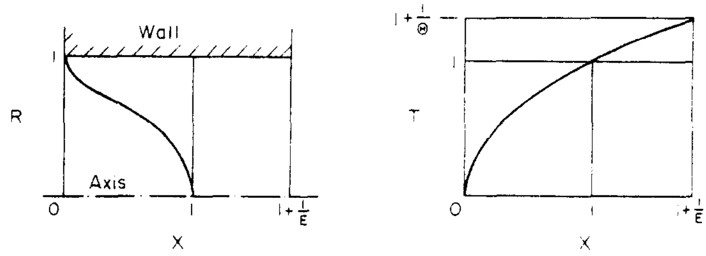

FIG. 4. Cylinder with thermal entry region ($X = 0$ to 1). Lower boundary of heated fluid on the R–X field. Wall temperature on the T–X field.

constant. The additional length of cylinder to provide Q_E is El. Figure 4 shows the extent of the heated fluid, and the wall temperature, in these conditions. Dimensionless axial distance is made unity at the upper boundary of the entry region, and dimensionless temperature unity when $t = \phi_{max}$.

The present values of u, l, Ra, t, θ and Nu are given the subscript "1". The alteration in Ra is due to increase of length from l to $l(1+E)$. Hence Ra is divided by $(1+E)$ to obtain Ra_1.

The mean wall temperature has been reduced from θ to θ_1, which is the mean height of the graph of T in Fig. 4. The area below this graph from $X = 0$ to $X = 1$ is termed f_E and is less than 1. The ratio of Nu_1 to Nu is given by

$$\frac{Nu_1}{Nu} = \frac{\theta}{\theta_1} = \frac{2E\Theta + 2\Theta + E + 1}{2f_E E\Theta + 2\Theta + 1}. \tag{31}$$

Values of f_E were all between 0.70 and 0.74.

Table 1 contains some values of Ra_1 and Nu_1. The relation of these is shown by a broken line on Fig. 5. This is not mathematically an exact solution, since the original distribution of t was compatible with the unchanged u_1 profile, and the new t_1 is not. Since t_1 is in fact too low for u_1, and lower temperatures imply a higher value of Nu, an exact overall heat-transfer relation is expected to run slightly below the curve (4) of Fig. 5. It is believed to be possible in principle to obtain a compatible u-profile and t-distribution by an iterative process between u and t in the sequence

$$u_1 \rightarrow t_1 \rightarrow u_2 \rightarrow t_2 \rightarrow \ldots. \tag{32}$$

This approach has been successfully applied to a similar system using a numerical method [5].

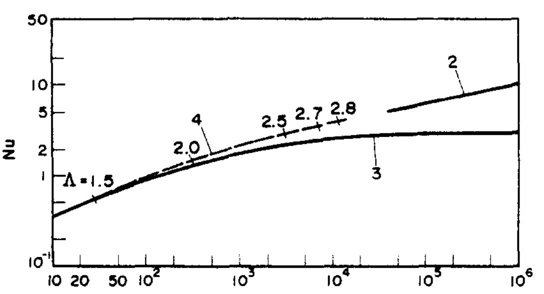

FIG. 5. Nusselt number against Rayleigh number: (2) Vertical wall: $Nu = 0.619 Ra^{0.20}$; (3) Cylinder with fully developed temperatures; (4) Inexact solution for cylinder with thermal entry region.

The open vertical channel with isothermal walls was analysed by Bodoia and Osterle [6] using a finite difference "marching" method. A uniform upward velocity was prescribed at entry, and a transverse velocity component included by means of the continuity equation. This method was applied to the present problem by Dyer [7]. As accurately as can be determined without numerical data, the Nusselt number results of Dyer coincide exactly with the curve (4) of Fig. 5, up to its termination. At higher values of Ra the results of Dyer merge with the relation $Nu = 0.67 Ra^{\frac{1}{5}}$.

CONCLUSIONS

Modification of the analysis for fully developed flow, by providing a thermal entry region in addition, has considerably extended its scope. The modified analysis covers conditions from small values of Rayleigh number up to a terminal point, when the thermal entry region exactly fills the cylinder, at Rayleigh number of about 16 000. The results as presented are approximate, but are capable of development to an exact state. The idealisation employed, that motion within the heated fluid is only in the axial direction, is not a very restrictive one for the conditions described.

Acknowledgements—This work is taken from the Ph.D. thesis of H. J. Hetherington [8]. It is gratefully acknowledged that the late Professor A. J. Ede encouraged the authors' research in this topic, and later acted as external examiner for the Ph.D. thesis of O. Ofi [5] on a related topic.

REFERENCES

1. W. Elenbaas, Heat dissipation of parallel plates by free convection, *Physica, s'Grav.* **9**, 1–27 (1942).
2. E. M. Sparrow and J. L. Gregg, Laminar free convection from a vertical plate with uniform heat flux, *Trans. Am. Soc. Mech. Engrs* **78**, 435–440 (1956).
3. J. Woodrow, Free convection in a heat generating fluid, Atomic Energy Research Estab. E/R 1267, Harwell, England (1953).
4. D. V. Wordsworth, Laminar free convection between heat producing vertical plates in a fluid, Atomic Energy Research Estab. E/R 1270, Harwell, England (1953).
5. O. Ofi, Natural convection heat transfer from the open vertical channel, Ph.D. Thesis, University of Dundee, Scotland (1972).
6. J. R. Bodoia and J. F. Osterle, The development of free convection between heated vertical plates, *J. Heat Transfer* **84C**(1), 40–44 (1962).
7. J. R. Dyer, The development of natural convection in a vertical circular duct, *Mech. Chem. Engng Trans. I.E. Aust.* **MC4**(1), 78–86 (1968).
8. H. J. Hetherington, Heat transfer from the vertical internal cylinder: free convection and radiation, Ph.D. Thesis, University of Edinburgh, Scotland (1966).

CONVECTION NATURELLE LAMINAIRE DANS UN CYLINDRE VERTICAL OUVERT AVEC FLUX THERMIQUE PARIETAL UNIFORME

Résumé—Une solution est obtenue pour l'écoulement établi d'un fluide à propriétés constantes, la température la plus basse étant du côté fluide. La solution est satisfaisante seulement aux faibles valeurs du nombre de Rayleigh, jusqu'à 20 environ. c'est à dire pour des cylindres relativement longs. Une deuxième solution présente de plus une région d'établissement thermique, pour l'application à des cylindres plus courts, au point où la région d'établissement rempli de cylindre. Ce problème n'a pas encore trouvé de solution mathématique exacte. Les nombres de Nusselt jusqu'au point extrême, à un nombre de Rayleigh d'environ 16 000, sont identiques à ceux obtenus par Dyer [6] avec différences finies.

LAMINARE NATÜRLICHE KONVEKTION IN EINEM OFFENEN VERTIKALEN ZYLINDER MIT GLEICHFÖRMIGER WANDWÄRMESTROMDICHTE

Zusammenfassung—Es wird eine Lösung angegeben für die voll ausgebildete Strömung eines Fluides mit konstanten Stoffwerten und für den Fall, daß die tiefste Temperatur der mittleren Fluidtemperatur entspricht. Dieser Fall ist nur für kleine Werte der Rayleigh-Zahl bis zu etwa 20 gültig, d.h. nur für relativ lange Zylinder. Eine zweite Lösung berücksichtigt zusätzlich den thermischen Einlaufbereich und kann auf kürzere Zylinder angewandt werden. Diese Lösung kann bis zu dem Grenzfall angewandt werden, in dem der Einlaufbereich den gesamten Zylinder umfaßt. Die Lösung konnte noch nicht mathematisch exakt formuliert werden. Die berechneten Nusselt-Zahlen sind bis zu Rayleigh-Zahlen von rund 16000 identisch mit den von Dyer [6] mit Hilfe der Methode finiter Differenzen ermittelten Werte.

ЛАМИНАРНАЯ ЕСТЕСТВЕННАЯ КОНВЕКЦИЯ ОТ ОТКРЫТОГО ВЕРТИКАЛЬНОГО ЦИЛИНДРА ПРИ ОДНОРОДНОМ ТЕПЛОВОМ ПОТОКЕ НА СТЕНКЕ

Аннотация — Получено решение для полностью развитого течения жидкости с постоянными свойствами, когда самое низкое значение температуры жидкости равно ее объемному значению. Такая ситуация реализуется только при низких значениях числа Релея (до 20), т. е. для цилиндров относительно большой длины. Обсуждается второе решение, которое строится с учетом теплового начального участка и относится к цилиндрам небольшой длины. Это решение, справедливое до точки, в которой заканчивается входная область, еще не получено в точном математическом виде. Числа Нуссельта на расстоянии до конечной точки при числах Релея, равных примерно 16 000, идентичны значениям, полученным Дайером с помощью конечно-разностного метода [6].